Appendix A

Gravity and general relativity: A classical field theory perspective

We devote this section for a brief overview of gravity as a classical field theory [1]. We start with Newtonian gravity as a classical field theory and introduce functional quantization in order to obtain Poisson's equation from a Lagrangian (Section A.1). We then introduce general relativity and the Einstein-Hilbert action from which the Einstein equation can be derived by functional differentiation (Section A.2).

A.1 Newtonian gravity

For weak fields, the attractive gravitational force F exerted by a massive source on a particle varies inversely with the square of the distance and is proportional to the masses of the source and the particle, i.e., $F \sim m_1 m_2 / r^2$. This behavior is well described when gravity, or the gravitational field, is considered to be a scalar field, $\Phi = \Phi\left(t, \vec{x}\right)$, a function of spacetime coordinates $\left(t, \vec{x}\right)$, which satisfies Poisson's

[1] This section is written for the reader who wishes to go through the rest of this thesis without so much excess mathematical baggage, but it is by no means a fateful representation of general relativity and classical field theory. A more honest introduction to classical field theory can be found in the last chapter of Goldstein's mechanics text [172]. Supplementary information on GR as a classical field theory could be found in Refs. [2, 173].

equation

$$\nabla^2 \Phi\left(t, \vec{x}\right) = 4\pi G \rho\left(t, \vec{x}\right) \tag{A.1}$$

where $\rho\left(t, \vec{x}\right)$ is the mass density of a source at time t at position $\vec{x}$ and G is Newton's gravitation constant. Obviously, when $\rho \sim \delta\left(\vec{x}\right)$, corresponding to a point source at the origin, Eq. (A.1) is satisfied by the Green's function $\Phi = -GM/r$, which incidentally is the gravitational potential (or gravitational potential energy per unit test mass) at a distance r away from the source. Poisson's equation thus captures the predictions of the inverse-square law and could very well be regarded as synonymous with Newtonian gravity.

However, rather than the field equations for a theory (e.g., Poisson's equation for Newtonian gravity), a more appealing theoretical device is the action functional or, simply, the action. The symmetries of a theory are transparent from the theory's action and the field equations could be derived by simply obtaining the functional derivative of the action with respect to the fields [2]. To illustrate this idea, we introduce the action functional $S\left[\Phi\right]$ of Newtonian gravity:

$$S\left[\Phi\right] = \int d^4x \left(\frac{1}{8\pi G} \nabla\Phi\left(t, \vec{x}\right) \cdot \nabla\Phi\left(t, \vec{x}\right) + \rho\left(t, \vec{x}\right)\Phi\left(t, \vec{x}\right)\right) \tag{A.2}$$

where $d^4x = dtd^3x$ where dt is an infinitesimal time increment and d^3x is the volume element, e.g., in spherical polar coordinates (r, θ, φ), $d^3x = drd\theta d\varphi r^2 \sin\theta$. For brevity, the dependence on spacetime coordinates $(t, \vec{x})$ will be suppressed in the following calculations. To obtain Poisson's equation from the action (A.2), we expand the action about the field Φ and retain only the linear order terms, as shown,

$$\begin{aligned} S\left[\Phi + \delta\Phi\right] =& \int d^4x \left(\frac{1}{8\pi G} \nabla\left(\Phi + \delta\Phi\right) \cdot \nabla\left(\Phi + \delta\Phi\right) + \rho\left(\Phi + \delta\Phi\right)\right) \\ =& S\left[\Phi\right] + \int d^4x \left(\frac{1}{4\pi G}\nabla\Phi \cdot \nabla\left(\delta\Phi\right) + \rho\delta\Phi\right) + O\left(\delta\Phi^2\right) \end{aligned} \tag{A.3}$$

[2] Moreover, the transition from classical to quantum field theory is straightforward from a theory's action.

where $S[\Phi]$ is given by Eq. (A.2) and $O(\delta\Phi^2)$ represents all of the terms of at least second order in the variation $\delta\Phi$. The functional derivative, $\delta S/\delta\Phi$, of the functional S with respect to the field Φ is defined as

$$S[\Phi + \delta\Phi] - S[\Phi] = \int d^4x \frac{\delta S}{\delta\Phi}\delta\Phi + \oint (\cdots) + O\left(\delta\Phi^2\right) \tag{A.4}$$

where $\oint(\cdots)$ corresponds to boundary terms, i.e., total derivatives. To obtain the functional derivative of Eq. (A.2), we simply perform integration by parts in Eq. (A.3) to put it into the form of Eq. (A.4), leading to

$$\begin{aligned} S[\Phi + \delta\Phi] - S[\Phi] = &\int d^4x \left(-\frac{1}{4\pi G}\nabla^2\Phi + \rho\right)\delta\Phi \\ &+ \int d^4x \nabla\cdot\left(\frac{\delta\Phi}{4\pi G}\nabla\Phi\right) + O\left(\delta\Phi^2\right) \end{aligned} \tag{A.5}$$

where the second integral $\int d^4x\nabla(\cdots)$ is the boundary term [3]. The functional derivative of the action (A.2) is therefore

$$\frac{\delta S}{\delta\Phi} = -\frac{1}{4\pi G}\nabla^2\Phi + \rho. \tag{A.6}$$

At this point, it becomes transparent that Poisson's equation (Eq. (A.1)) emerges by demanding that the functional derivative vanishes. From the classical field theory point-of-view, the field Φ satisfying Poisson's equation is on-shell, meaning that it obeys the classical field equation or, alternatively, that it extremizes the theory's action. The rest of the predictions of Newtonian gravity follow from solving Poisson's equation.

A.2 General relativity

For strong fields, e.g., in the vicinity of neutron stars and black holes, it is well-known that one must use the general theory of relativity (GR) to describe the

[3]Recall Gauss' law, $\int_V d^3x\nabla\cdot\vec{A} = \oint_{S=\partial V} d\sigma\ \hat{n}\cdot\vec{A}$, where $\vec{A}$ is an arbitrary vector, d^3x is the volume element of V, $d\sigma$ and $\hat{n}$ are the surface element and unit normal to the boundary $S = \partial_V$ of volume V. Boundary terms do not contribute to the classical field equations; however, they are intimately related to the well-posedness of a variational problem [173].

gravitational field. Instead of a scalar field, however, in GR, gravity is the spacetime curvature encoded in the symmetric tensor $g_{ab}(x)$, called the metric, where $x = (ct, \vec{x})$ stand for the spacetime coordinates and the indices $\{a, b, \cdots\}$ run over $\{0, 1, 2, 3\}$ with index 0 corresponding to time component and $\{1, 2, 3\}$ to spatial components. This metric is coupled to matter sources, just as the scalar field Φ is coupled to ρ in Eq. (A.1), via the Einstein equation

$$G_{ab} = \frac{8\pi G}{c^4} T_{ab} \tag{A.7}$$

where G_{ab} is the so-called Einstein tensor, T_{ab} is the matter sources' stress-energy tensor (SET), and c is the speed of light. The direct coupling with the SET means that the gravity is sourced by energy, rather than by only the mass density as in Newtonian gravity, and that all matter, including light, should be deflected by gravitational fields – this is the *equivalence principle*. Furthermore, it should be emphasized that the Einstein tensor is a second-order nonlinear differential operator on the metric, i.e., $G_{ab} = G_{ab}[g_{cd}, \partial g_{cd}, \partial^2 g_{cd}]$. In the simplest words, this means that gravity sources itself [4]. To see this, the Einstein tensor can be expressed as

$$G_{ab} = R_{ab} - \frac{R}{2} g_{ab} \tag{A.8}$$

where $R_{ab} = R^c{}_{acb}$ and $R = g^{ab} R_{ab}$ are the Ricci tensor and Ricci scalar, respectively, and $R^a{}_{bcd}$ is Riemann tensor given by

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc} \tag{A.9}$$

where Γ^a_{bc} are the Christoffel connections,

$$\Gamma^a_{bc} = \frac{g^{ad}}{2}\left(g_{db,c} + g_{dc,b} - g_{bc,d}\right). \tag{A.10}$$

In the above expressions, $g_{ab,c} = \partial g_{ab}/\partial x^c$ and $\Gamma^a_{bc,d} = \partial \Gamma^a_{bc}/\partial x^d$. Although a much more complex (but theoretically and phenomenologically richer) coupled differential equation for gravity, for weak fields, $g_{ab} = \mathrm{diag}\left(-1 + 2\Phi, 1 - 2\Phi, r^2, r^2 \sin^2\theta\right)$,

[4] This is in stark contrast with Poisson's equation which implies that the gravity vanishes everywhere in vacuum; in other words, $\Phi = 0$ for all t and $\vec{x}$ if $\rho = 0$, assuming that there are no artificial boundary conditions (typically exploited in electrostatics). In GR, even if $T_{ab} = 0$, the nonlinearity of the Einstein equation guarantees nontrivial vacuum solutions, e.g., black holes.

Eq. (A.7) reduces to Eq. (A.1); the Einstein equation is the completion of Poisson's equation for strong fields.

As a classical field theory, GR is described by the action:

$$S\left[g_{ab}\right] = \int d^4x\sqrt{-g}\left(\frac{c^4}{16\pi G}R\right) + S_M\left[\Psi, g_{ab}\right] \tag{A.11}$$

where g is the determinant of g_{ab}, $d^4x\sqrt{-g}$ is the invariant spacetime volume element, and $S_M\left[\Psi, g_{ab}\right]$ is the functional describing the matter fields Ψ and their coupling with gravity. The first term in the right hand side of Eq. (A.11) is the Einstein-Hilbert gravitational action. To obtain the Einstein equation (Eq. (A.7)), we vary the action (Eq. (A.11)) with respect to the metric, g_{ab}, or equivalently, the inverse metric g^{ab}, defined by $g_{ac}g^{cb} = \delta_a^b$ where δ_a^b is the Kronecker delta. Using some well-known identities,

$$R = R_{ab}g^{ab} \tag{A.12}$$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab} \tag{A.13}$$

$$g^{ab}\delta R_{ab} = \nabla_a I^a \tag{A.14}$$

$$\nabla_a g_{bc} = 0, \tag{A.15}$$

where I^a is some vector [5], and ∇_a is the covariant derivative with respect to the metric g_{ab} [6], then we obtain

$$\begin{aligned}
\delta S =& \frac{c^4}{16\pi G}\int d^4x\left(\left(\delta\sqrt{-g}\right)R + \sqrt{-g}R_{ab}\delta g^{ab} + \sqrt{-g}g^{ab}\delta R_{ab}\right) + \delta S_M \\
=& \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\left(R_{ab} - \frac{R}{2}g_{ab}\right)\delta g^{ab} + \frac{c^4}{16\pi G}\int d^4x\nabla_a\left(\sqrt{-g}I^a\right) + \delta S_M \\
\delta S =& \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}G_{ab}\delta g^{ab} + \frac{c^4}{16\pi G}\int d^4x\nabla_a\left(\sqrt{-g}I^a\right) + \delta S_M
\end{aligned} \tag{A.16}$$

[5] Specifically, $I^c = g_{ab}\left(\nabla^c\delta g^{ab}\right) - \left(\nabla_a\delta^a c\right)$, but the necessary observation here is that the variation of the Ricci tensor is a total derivative.

[6] The covariant derivative ∇_a is the curved space generalization of the partial derivative ∂_a to make the theory invariant under general coordinate transformations. It satisfies nearly all of the properties of the partial derivative, such as linearity and Leibnitz rule, but its most notable feature is that covariant derivatives do not commute, e.g., $\left(\nabla_b\nabla_a - \nabla_a\nabla_b\right)A^c = -R^c{}_{dab}A^d$ for an arbitrary vector A^a.

where $\delta S = S\left[g_{ab} + \delta g_{ab}\right] - S\left[g_{ab}\right]$ and we have dropped the terms $O\left(\delta g^2\right)$. Noting that the second term in the last line of Eq. (A.16) is a total derivative, the functional derivative of the Einstein-Hilbert action plus the matter action $S_M\left[\Psi, g_{ab}\right]$ is given by

$$\frac{\delta S}{\delta g^{ab}} = \sqrt{-g}\frac{c^4}{16\pi G}G_{ab} + \frac{\delta S_M}{\delta g^{ab}}. \tag{A.17}$$

Finally, defining the matter's SET, T_{ab}, in terms of the action $S_M\left[\Psi, g_{ab}\right]$ by

$$\frac{\delta S_M}{\delta g^{ab}} = -\frac{\sqrt{-g}}{2}T_{ab}, \tag{A.18}$$

then the Einstein equation (Eq. (A.7)) is obtained by setting Eq. (A.17) to zero. The rest of the predictions of GR follow by solving the Einstein equation, or for the on-shell metric, g_{ab}.

Appendix B

Variation of quadratic and cubic Horndeski action

In this section, we explicitly write down the steps performed to obtain the contributions of the quadratic and cubic sectors of Horndeski theory,

$$S\left[g_{ab},\phi\right]=\int d^4x\left(G_2\left(\phi,X\right)-G_3\left(\phi,X\right)\Box\phi\right) \tag{B.1}$$

to the gravitational field equations. In Eq. (B.1), ϕ is the scalar field, $\Box\phi=\nabla^a\nabla_a\phi$ and $X=-g^{ab}\left(\partial_a\phi\right)\left(\partial_b\phi\right)/2$ is the scalar field's kinetic density. For brevity, in what follows, we shall use the notation

$$G_{iX} = \frac{\partial G_i}{\partial X} \tag{B.2}$$

$$G_{i\phi} = \frac{\partial G_i}{\partial \phi} \tag{B.3}$$

$$\phi_a = \partial_a\phi \tag{B.4}$$

$$\phi_{ab} = \nabla_a\nabla_b\phi \tag{B.5}$$

where $i=2,3$. It is also useful to note that $\phi_{ab}=\phi_{ba}$. For reference, the full Horndeski theory and its covariant field equations can be found in Ref. [93].

B.1 Variation with respect to the metric

The variation of Eq. (B.1) with respect to the inverse metric leads to

$$\begin{aligned}\delta S = &\int d^4x \left(-\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}\right)(G_2 - G_3\Box\phi) \\ &+ \int d^4x\sqrt{-g}\left[\delta G_2 - \delta G_3\Box\phi - G_3\delta\Box\phi\right].\end{aligned} \tag{B.6}$$

Noting that the Horndeski potentials, $G_i(\phi, X)$, depend on the metric g^{ab} only through the kinetic density $X \sim g^{ab}\phi_a\phi_b$, then

$$\delta G_i = G_{iX}X_{ab}\delta g^{ab} \tag{B.7}$$

where

$$X_{ab} = -\frac{\phi_a\phi_b}{2}. \tag{B.8}$$

Also, the variation of $\Box\phi$ with respect to the metric can be obtained as follows,

$$\begin{aligned}\delta\Box\phi &= \delta\left(g^{ab}\nabla_a\nabla_b\phi\right) \\ &= \delta g^{ab}\nabla_a\nabla_b\phi + g^{ab}\delta\left(\nabla_a\nabla_b\phi\right) \\ &= \delta g^{ab}\nabla_a\nabla_b\phi + g^{ab}\delta\left(\partial_a\partial_b\phi - \Gamma^c_{ab}\partial_c\phi\right) \\ &= \delta g^{ab}\nabla_a\nabla_b\phi - g^{ab}\delta\Gamma^c_{ab}\partial_c\phi \\ \delta\Box\phi &= \delta g^{ab}\phi_{ab} + \frac{\phi_c}{2}\left(2\nabla_b\left(\delta g^{bc}\right) - g_{ij}\nabla^c\left(\delta g^{ij}\right)\right),\end{aligned} \tag{B.9}$$

where to get to the last line we have used the identity

$$\delta\Gamma^c_{ab} = -\frac{1}{2}\left[g_{da}\nabla_b\left(\delta g^{dc}\right) + g_{db}\nabla_a\left(\delta g^{dc}\right) - g_{ai}g_{bj}\nabla^c\left(\delta g^{ij}\right)\right]. \tag{B.10}$$

The variation of the action thus can be written as

$$\begin{aligned}\delta S = &\int d^4x\sqrt{-g}\Bigg[-\frac{1}{2}g_{ab}(G_2 - G_3\Box\phi) + G_{2X}X_{ab} - G_{3X}X_{ab}\Box\phi - G_3\phi_{ab}\Bigg]\delta g^{ab} \\ &+ \int d^4x\sqrt{-g}\Bigg[-G_3\frac{\phi_c}{2}\left(2\nabla_b\left(\delta g^{bc}\right) - g_{ij}\nabla^c\left(\delta g^{ij}\right)\right)\Bigg].\end{aligned} \tag{B.11}$$

Performing integration by parts on the second line of the above expression,

$$
\begin{aligned}
&-\int d^4x\sqrt{-g}G_3\phi_c\nabla_b\left(\delta g^{bc}\right)+\int d^4x\sqrt{-g}\frac{G_3}{2}\phi_c g_{ij}\nabla^c\left(\delta g^{ij}\right)\\
&=\int d^4x\sqrt{-g}\nabla_b\left(G_3\phi_c\right)\delta g^{bc}-\frac{1}{2}\int d^4x\sqrt{-g}\nabla^c\left(G_3\phi_c\right)g_{ij}\delta g^{ij}+\int\nabla_a\left(\cdots\right),
\end{aligned}
\tag{B.12}
$$

where $\int\nabla_a\left(\cdots\right)$ are boundary terms, we obtain

$$
\begin{aligned}
\delta S=\int d^4x\sqrt{-g}\Big[&-\frac{1}{2}g_{ab}\left(G_2-G_3\Box\phi\right)+G_{2X}X_{ab}-G_{3X}X_{ab}\Box\phi\\
&-G_3\phi_{ab}+\nabla_a\left(G_3\phi_b\right)-\frac{1}{2}g_{ab}\nabla^c\left(G_3\phi_c\right)\Big]\delta g^{ab}+\int\nabla_a\left(\cdots\right).
\end{aligned}
\tag{B.13}
$$

The last terms in the square bracket can be expressed as

$$\nabla_a\left(G_3\phi_b\right) = G_3\phi_{ab}+G_{3a}\phi_b \tag{B.14}$$

$$\nabla^c\left(G_3\phi_c\right) = G_{3a}\phi^a+G_3\Box\phi. \tag{B.15}$$

Finally, we obtain the variation of the quadratic and cubic Horndeski action as

$$
\begin{aligned}
\delta S=\int d^4x\sqrt{-g}\Big[&-\frac{1}{2}g_{ab}G_2+G_{2X}X_{ab}-G_{3X}X_{ab}\Box\phi\\
&+G_{3(a}\phi_{b)}-\frac{1}{2}g_{ab}G_{3c}\phi^c\Big]\delta g^{ab}+\int\nabla_a\left(\cdots\right).
\end{aligned}
\tag{B.16}
$$

The functional derivative of S with respect to δg^{ab} is therefore

$$\frac{\delta S}{\delta g^{ab}}=\sqrt{-g}\left[-\frac{1}{2}g_{ab}G_2+G_{2X}X_{ab}-G_{3X}X_{ab}\Box\phi+G_{3(a}\phi_{b)}-\frac{1}{2}g_{ab}G_{3c}\phi^c\right] \tag{B.17}$$

so that the quadratic and cubic Horndeski sectors' contribution to the Einstein equation is via the scalar field's stress-energy tensor (see Eq. (A.18))

$$T^{(\phi)}_{ab}=g_{ab}G_2-2G_{2X}X_{ab}+2G_{3X}X_{ab}\Box\phi-2G_{3(a}\phi_{b)}+g_{ab}G_{3c}\phi^c. \tag{B.18}$$

B.2 Variation with respect to the scalar field

In scalar-tensor theories such as Horndeski theory, the scalar field, ϕ, brings its own field equation. This can be obtained by varying the action with respect to ϕ, i.e., calculate $\delta S=S\left[\phi+\delta\phi\right]-S\left[\phi\right]$.

To obtain the scalar field equation for the quadratic and cubic sectors of Horndeski theory, we therefore vary Eq. (B.1) with respect to ϕ. For the quadratic sector, we simply note that

$$G_2[\phi+\delta\phi, X+\delta X] = G_2 + G_{2\phi}\delta\phi + G_{2X}\delta X \tag{B.19}$$

where

$$\delta X = -g^{ab}\phi_a\left(\partial_b\delta\phi\right). \tag{B.20}$$

In contrast with the variation with respect to the metric, δg^{ab}, both of the arguments of the potentials $G_i\left(\phi, X\right)$ should now be expanded in terms of the variation $\delta\phi$. Thus, the variation of the action from the quadratic sector is

$$\begin{aligned}\delta S_2 &= \int d^4x\sqrt{-g}\left(G_{2\phi}\delta\phi + G_{2X}\delta X\right) \\ &= \int d^4x\sqrt{-g}\left(G_{2\phi}\delta\phi - G_{2X}g^{ab}\phi_a\left(\partial_b\delta\phi\right)\right)\end{aligned} \tag{B.21}$$

where we have dropped the $O\left(\delta\phi^2\right)$ terms. Performing integration by parts on the second term, we obtain

$$\delta S_2 = \int d^4x\sqrt{-g}\left(G_{2\phi} + g^{ab}\nabla_b\left(G_{2X}\phi_a\right)\right)\delta\phi + \int\nabla_a\left(\cdots\right). \tag{B.22}$$

For the cubic sector, we note that

$$\begin{aligned}-\delta\left[G_3\Box\phi\right] &= -\left(\delta G_3\right)\Box\phi - G_3\Box\left(\delta\phi\right) \\ &= -\left(G_{3\phi}\delta\phi + G_{3X}\delta X\right)\Box\phi - G_3\Box\left(\delta\phi\right).\end{aligned} \tag{B.23}$$

Perform integration by parts once on the second term and twice on the third term, we obtain

$$\delta S_3 = \int d^4x\sqrt{-g}\left[-G_{3\phi} - g^{ab}\nabla_b\left[G_{3X}\phi_a\Box\phi\right] - \Box G_3\right]\delta\phi + \int\nabla_a\left(\cdots\right). \tag{B.24}$$

Finally, by adding Eqs. (B.22) and (B.24), we obtain the variation of Eq. (B.1) with respect to the scalar field [1]:

$$\delta S = \int d^4x\sqrt{-g}\Big[G_{2\phi} + g^{ab}\nabla_b\left(G_{2X}\phi_a\right) \\ - G_{3\phi} - g^{ab}\nabla_b\left[G_{3X}\phi_a\Box\phi\right] - \Box G_3\Big]\delta\phi + \int \nabla_a\left(\cdots\right). \tag{B.25}$$

The functional derivative, $\delta S/\delta\phi$, is therefore the expression inside the square brackets of the first integral. By setting this to zero, we end up with the field equation satisfied by ϕ:

$$\left[G_{2\phi} + g^{ab}\nabla_b\left(G_{2X}\phi_a\right)\right] + \left[-G_{3\phi} - g^{ab}\nabla_b\left[G_{3X}\phi_a\Box\phi\right] - \Box G_3\right] = 0. \tag{B.26}$$

As a humble check, for a minimally coupled scalar field ϕ in a potential $V(\phi)$, or quintessence models, $G_2 = X - V(\phi)$ and $G_3 =$ constant, Eq. (B.26) reduces to the Klein-Gordon equation, $\Box\phi = -V'(\phi)$.

[1] An assumption here is that the scalar field does not directly enter the matter action $S_M[\Psi, g_{ab}]$, i.e., $\delta S_M/\delta\phi = 0$. This is supported by the equivalence principle which must be valid even for scalar-tensor theories.